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Proof of a New Circle Method of Goldbach's Conjecture

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Abstract: In this paper, a novel circle method is introduced which, compared to previous approaches, eliminates the need to explicitly estimate the prime-variable triangle sum on the residual interval [1]. By employing the Fourier series to express the summation formula, we estimate the triangle sum on the residual interval. At the same time, the concept of the intersection set is introduced. Using this concept, we recalculate the estimated values on both the main and residual intervals, thereby forming a new circle method. This new approach focuses on proving that the main value of the solution count is equivalent to its value on the main interval.

Keywords: New circle method; Exception module; Real zero distribution; Asymptotic formula of solution number

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Let N be a sufficiently large even number, N_3, P, M_1, a_1, b_2 be positive integer, m_{01} be natural number.Assume that the following basic relationship holds

$$a_1 > 3b_2 , b_2 \geq 3 , \tau_0 = e^{\frac{6}{5}(\log N)^{\frac{7}{12}}} , \tau_1 = \log^{a_1} N , D_0 = 2[\tau_0\tau_1] ,$$

$$P = \sum_{m=0}^{M_0} P_m , N_3 < P_0 \leq 2N_3 , (N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right) < P_m \leq 2(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)$$

$$N_3 = \left[\frac{N}{(\tau_0\tau_1)^2}\right], M_0 = \left[\frac{\log\left(\log\frac{N_3}{(D_0)^6}\right)}{\log 2}\right], M_1 = \left[\frac{\tau_0\tau_1}{\log^{b_2} N}\right] , M_1 < m_{01} \leq 2M_1$$

$$E(dP) = \frac{1}{dPD_0} , 1 \leq d \leq D_0 \tag{1}$$

For each value $y \in \left[-\frac{1}{\tau}, 1-\frac{1}{\tau}\right)$ could write as

$$y = \frac{a}{q} + z , (a,q) = 1 , 1 \leq q \leq \tau , |z| \leq \frac{1}{q\tau}$$

Where

$$\tau = \frac{N}{(\tau_0\tau_1)^{2.8}}$$

Definite $I_1(\tau_0)$ be the main interval, if $y \in I_1(\tau_0)$,then

$$y = \frac{a}{q} + z, 0 \le a < q, (a,q) = 1, 1 \le q < Q_1, |z| \le \frac{1}{q(\tau_0\tau_1)\tau}$$

Where

$$Q_1 = e(D_0)^6, \quad q(\tau_0\ \tau_1) = \begin{cases} q, & q < \tau_0\tau_1 \\ \tau_0\tau_1, & q \ge \tau_0\tau_1 \end{cases}$$

The remaining interval interval is

$$I_2(\tau_0) = \left[-\frac{1}{\tau}, 1 - \frac{1}{\tau}\right) \backslash I_1(\tau_0)$$

For each valu α in the interval,the prime variable function can be expressed as:

$$S(\alpha) = \sum_{p=N_2}^{N} e^{2i\pi\alpha p}$$

Where

$$N_2 = \frac{N}{\log^{2b_2} N}$$

Prime variable equation $p_1 + p_2 = N$ decompose on the $I_1(\tau_0)$.Formula (3)~(12) proof can be see chapter 11 of reference 4.We have

$$D1(N) = \sum_{q \le Q_1} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\frac{a}{q} - \frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{a}{q} + \frac{1}{q(\tau_0\tau_1)\tau}} S^2(\alpha) e(-N\alpha) d\alpha = \sum_{j=1}^{6} D_{1j}(N)$$

(2)

where

$$D_{1,1}(N) = \sum_{q \le Q_1} \frac{\mu^2(q)}{\phi^2(q)} C_q(-N) \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} T^2(z) e(-Nz) dz,$$

$$T(z) = \sum_{n=N_2}^{N} \frac{e^{2\pi i n z}}{\log n}$$

(3)

$$D_{12}(N) = 2 \sum_{q \le Q_1} \frac{\mu(q)}{\phi^2(q)} \sum_{\chi_q} \tau\left(\bar{\chi}\right) G_{\chi}(-N) \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} T(z) W(z,\chi) e(-Nz) dz,$$

$$W(z,\chi) = \sum_{p=N_2}^{N} \chi(p) e^{2\pi i p z}$$

(4)

$$D_{13}(N) = \sum_{q \le Q_1} \frac{1}{\phi^2(q)} \sum_{\chi_q} \sum_{\chi_q'} \tau\left(\bar{\chi}\right) \tau\left(\bar{\chi}'\right) G_{\chi\chi'}(-N) \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} W(z,\chi) W(z,\chi') e(-Nz) dz$$

(5)

$$D_{1,4}(N) = \sum_{\substack{q \le Q_1 \\ q_0 | q}} \frac{C_q(-N)}{\phi^2(q)} \tau^2\left(\chi_q^0 \tilde{\chi}\right) \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} \left(\tilde{T}(z)\right)^2 e(-Nz) dz,$$

$$\tilde{T}(z) = \sum_{n=N_2}^{N} n^{\beta_0 - 1} \frac{e^{2\pi i n z}}{\log n}$$

(6)

$$D_{15}(N)=2\sum_{\substack{q\le Q_1\\ q_0|q}}\frac{\mu(q)}{\phi^2(q)}\sum_{\chi_q}\tau\left(\chi_q^0\tilde{\chi}\right)G_{\chi_q^0\tilde{\chi}}(-N)\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}T(z)\tilde{T}(z)e(-Nz)dz \tag{7}$$

$$D_{16}(N)=2\sum_{\substack{q\le Q_1\\ q_0|q}}\frac{\tau\left(\chi_q^0\chi\right)}{\phi^2(q)}\sum_{\chi_q}G_{\chi\tilde{\chi}}(-N)\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\tilde{T}(z)W(z,\chi)e(-Nz)dz \tag{8}$$

Estimation of $D_{1j}(N)$, $j=2,3,6$

$$\left|D_{1j}(N)\right|\le 64\frac{\sqrt{N}}{\log N}\frac{N}{\phi(N)}W+O\left(\frac{N}{(Q_1)^{10}}\right)\quad J=2,6 \tag{9}$$

$$\left|D_{13}(N)\right|\le 64\frac{1}{\log N}\frac{N}{\phi(N)}W^2+O\left(\frac{N}{(Q_1)^{10}}\right) \tag{10}$$

$$W=\sum_{d\le Q_1}\sum_{\chi_d}\cdot\left(\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\left|\sum_{n=N_2}^{N}\sum_{|\operatorname{Im}\rho|\le Q_1^{20}}n^{\rho-1}\frac{e^{2\pi inz}}{\log n}\right|^2dz\right)^{\frac{1}{2}}$$
$$\prec\prec\frac{\sqrt{N}}{e^{\frac{c_{10}\log N}{\log Q_1}}} \tag{11}$$

Formula proof can be see reference 4,chapter 4 theorem 2 and chapter 10 lemma 11 take $T=(Q_1)^{20}$ and c_9 be a constant in the lemma 11 and let $c_{10}=\vartheta c_9$, $0.97\le\vartheta\le 0.99$ .In this paper,mathematical symbols inherit this literature..

It can be obtained by formula (6)

$$D_{1,4}(N)=\tilde{\chi}(-1)\frac{q_0C_{q_0}(-N)}{\phi^2(q_0)}\sum_{k\le\frac{Q_i^2}{q_0}}\tilde{\chi}^2(k)\frac{\mu^2(k)C_k(-N)}{\phi^2(k)}\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\left(\tilde{T}(z)\right)^2e(-Nz)dz \tag{12}$$

Lemma 1.

Let $\alpha=\frac{a}{q}+\frac{\theta}{q^2}$, $(a,q)=1$, $q\ge 1$, $|\theta|\le 1$, then for arbitrarily $\beta, U>0$, and integer $N_{\cdot 1}\ge 1$,we have

$$\sum_{n=1}^{N_1}\min\left(U,\frac{1}{<\alpha n+\beta>}\right)\le 6\left(\frac{N_1}{q}+1\right)(U+q\log q)$$

Proof see chapter 5 lemma 6 of reference 3.

Lemma.2 Let $\tau\ge 1$, α be a real number, P be a positive integer, $P_1\mid P$ .Then exist coprime integers a and qP_1 , $1\le q\le\tau$,makes

$$\left|\alpha-\frac{a}{qP_1}\right|\le\frac{1}{qP\tau}$$

Proof. Without loss of generality,it can be assumed that $0\le\alpha<1$,conside $\{\alpha Pm\}$, $m=0,1,\dots,[\tau]$.

You must find two integers $m_1>m_2$,makes

$(\{\alpha Pm_1\}-\{\alpha Pm_2\})\le\frac{1}{\tau}$,namely $|\alpha q_1-b_1|\le\frac{1}{\tau}$, where $0<(m_1-m_2)P=q_1\le\tau P$, $b_1=[\alpha Pm_1]-[\alpha Pm_{2.}]$.This is the conclusion of exiting lemma.

According to equation (2),it can be assumed that

$$J_0(N)=\int_0^1 S^2(\alpha)e^{-2i\pi\alpha N}d\alpha \quad and \quad J_0(N,I_1(\tau_0))=D1(N)=\int_{I_1(\tau_0)} S^2(\alpha)e^{-2i\pi\alpha N}d\alpha$$

$$J_0(N,m_{01}dP)=\int_0^1 S^2(\alpha)e^{-2i\pi\alpha(N-m_{01}dP)}d\alpha \tag{13}$$

Theorem 1(New Circle Method).

Given the parameters same as above (1)~(13).then

$$J_0(N)=J_0(N,I_1(\tau_0))+O\left(\frac{N}{\sqrt{\tau_0\tau_1}}\right)$$

In order to prove the theorem 1,we need to proof lemma 3.

Define the following related functions

$$P1(x)=\sum_{n=1}^{\infty}\frac{Sin(2n\pi x)}{n\pi},\ \frac{1}{2}+[x]-x=P1(x),\ x\ne 0,\pm1,\pm2,...,\quad Euler\ \ function$$

$$F(P,m_{01},N)=\sum_{d=1}^{D_0}\sum_{u=0}^{dP-1}\int_{-E(dP)}^{E(dP)}S^2\left(\frac{u}{dP}+z\right)e^{-2i\pi\left(\frac{u}{dP}+z\right)(N-m_{01}dP)}dz$$

$$F(P1(Py),m_{01},N)=\sum_{d=1}^{D_0}\int_0^1\{P1(dPy-dPE(dP))-P1(dPy+dPE(dP))\}S^2(y)e^{-2i\pi(N-m_{01}dP)y}dy \tag{14}$$

and following sets

$$I_3(dP)\equiv\frac{x}{dP}+z\ ,\ x=0,...,dP-1\ ,\ -E(dP)\le z\le E(dP)$$

$$U(P)\equiv\bigcup_{d=1}^{D_0}[-E(dP),1-E(dP)]$$

Then by lemma 2,

$$\sum_{d=1}^{D_0}I_3(dP)\supseteq U$$

Quote the following symbols

$$S_1\oplus=\sum_{m_{01}=M_1+1}^{2M_1}\sum_{N_3<P_0\le 2N_3}\prod_{m=1}^{M_0}\sum_{(N_3)^{\frac{1`}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)<P_m\le 2(N_3)^{\frac{1`}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)}\frac{1}{N_3\prod_{m=1}^{M_0}(N_3)^{\frac{1`}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)M_1}\ ,$$

Thus

$$S_1\oplus F(P,m_{01},N)=S_1\oplus F11(((d_PP)_y,m_{01},N),U(P))+S_1\oplus F2((P,m_{01},N))$$

Where

$$F1(((d_PP)_y,m_{01},N),U(P))=\int_{U(P)}S^2(y)e^{-2i\pi y(N-m_{01}(d_PP)_y)y}dy\ ,\quad y=\frac{x}{dP}+z\ ,\quad |z|\le\frac{1}{dPD_0},\quad 0<d\le D_0$$

$$F2((P,m_{01},N))$$

$$=\left(\sum_{d=2}^{\frac{D_0}{2}}\sum_{j=1}^{d}+\sum_{d=\frac{D_0}{2}+1}^{D_0}\sum_{j=2d-D_0}^{d}\right)\sum_{t=0}^{P-1}\left\{\left\{\int_{E(dP)-DIS(j,d)}^{E(dP)}S^2\left(\frac{a(d,j)}{dP}+\frac{t}{P}+z\right)e^{-2i\pi\left(\frac{a(d,j)}{d}+\frac{t}{P}+z\right)(N-m_{01}dP)}dz+\int_{-E((D_0-d+j)P)}^{-(E((D_0-d+j)P)-DIS(j,d))}S^2\left(\frac{b(d,j)}{(D_0-d+j)P}+\frac{t}{P}+z\right)e^{-2i\pi\left(\frac{b(d,j)}{(D_0-d+j)}+\frac{t}{P}+z\right)(N-m_{01}(D_0-d+j)P)}dz\right\}\right.$$

$$\left.+\left\{\int_{-E(dP)}^{-(E(dP)-DIS(j,d))}S^2\left(-\frac{a(d,j)}{dP}+\frac{t}{P}+z\right)e^{-2i\pi\left(-\frac{a(d,j)}{d}+\frac{t}{P}+z\right)(N-m_{01}dP)}dz+\int_{(E((D_0-d+j)P)-DIS(j,d))}^{E((D_0-d+j)P)}S^2\left(-\frac{b(d,j)}{(D_0-d+j)P}+\frac{t}{P}+z\right)e^{-2i\pi\left(-\frac{b(d,j)}{(D_0-d+j)}+\frac{t}{P}+z\right)(N-m_{01}(D_0-d+j)P)}dz\right\}\right\}$$

$$,\quad a(d,j)(D_0-d+j)-b(d,j)d=-1,\quad a(d,j)<d\ ,\ b(d,j)<(D_0-d+j),$$

$$DIS(j,d)=\frac{j}{2d(D_0-d+j)D_0P}$$

(15)

Correspondingly ,there are

$$
\begin{aligned}
&F2((P1(Py),m_{01},N))\\
&=\left(\sum_{d=2}^{\frac{D_0}{2}}\sum_{j=1}^{d}+\sum_{d=\frac{D_0}{2}+1}^{D_0}\sum_{j=2d-D_0}^{d}\right)\left\{\int_0^1\left\{P1\left(Py-\frac{a(d,j)}{d}-PE(dP)\right)-P1\left(Py-\frac{a(d,j)}{d}-P(E(dP)-DIS(j,d))\right)\right\}S^2(y)e^{-2i\pi(N-m_{01}dP)y}dy\right.\\
&\left.+\int_0^1\left\{P1\left(Py-\frac{b(d,j)}{(D_0-d+j)}+P(E((D_0-d+j)P)-DIS(j,d))\right)-P1\left(Py-\frac{b(d,j)}{(D_0-d+j)}+PE((D_0-d+j)P)\right)\right\}S^2(y)e^{-2i\pi(N-m_{01}(D_0-d+j)P)y}dy\right\}\\
&+\left(\sum_{d=2}^{\frac{D_0}{2}}\sum_{j=1}^{d}+\sum_{d=\frac{D_0}{2}+1}^{D_0}\sum_{j=2d-D_0}^{d}\right)\left\{\int_0^1\left\{P1\left(Py+\frac{a(d,j)}{d}+P(E(dP)-DIS(j,d))\right)-P1\left(Py+\frac{a(d,j)}{d}+PE(dP)\right)\right\}S^2(y)e^{-2i\pi(N-m_{01}dP)y}dy\right.\\
&\left.+\int_0^1\left\{P1\left(Py+\frac{b(d,j)}{(D_0-d+j)}-PE((D_0-d+j)P)\right)-P1\left(Py+\frac{b(d,j)}{(D_0-d+j)}-P(E((D_0-d+j)P)-DIS(j,d))\right)\right\}S^2(y)e^{-2i\pi(N-m_{01}(D_0-d+j)P)y}dy\right\}
\end{aligned}
$$

(16)

then we have

Lemma 3. Given the parameters in (1), (14),(15),(16) .Let

$$S_1\oplus C_0(\tau_0)=S_1\oplus\sum_{d=1}^{D_0}\frac{e^{2\pi i m_{01}dPE(dP)}-e^{-2\pi i m_{01}dPE(dP)}}{2\pi i m_{01}}$$

$$S_1\oplus C_1(\tau_0)=2\left(\sum_{d=2}^{\frac{D_0}{2}}\sum_{\substack{j=1\\(d,(D_0-d+j))=1}}^{d}+\sum_{d=\frac{D_0}{2}+1}^{D_0}\sum_{j=2d-D_0}^{d}\right)\frac{j}{d(D_0-d+j)D_0}+O\left(\frac{1}{\log^{b_2}N}\right)$$

Then

$$
\begin{aligned}
&S_1\oplus(C_0(\tau_0)-C_1(\tau_0))(J_0(N)-J_0(N,I_1(\tau_0)))\\
&=S_1\oplus F1(((d_PP)_y,m_{01},N),U(P)\cap I_2(\tau_0))+O\left(\frac{N}{\sqrt{\tau_0\tau_1}}\right)
\end{aligned}
$$

Proof. With Euler summation formula we have

$$
\begin{aligned}
F((P,m_{01},N))&=\sum_{d=1}^{D_0}\frac{2}{D_0}J(N,m_{01}dP)-\sum_{d=1}^{D_0}\int_{-E(dP)}^{E(dP)}\int_0^{dP}\left[S^2\left(\frac{t}{dP}+z\right)e^{-2\pi i\left(\frac{t}{dP}++z\right)(N-m_{01}dP)}\right]_t^{/}P1(t)dtdz\\
&=\sum_{d=1}^{D_0}\frac{2}{D_0}J(N,m_{01}dP)-\sum_{d=1}^{D_0}\int_0^1\{P1(dPy-dPE(dP))-P1(dPy+dPE(dP))\}S^2(y)e^{-2i\pi(N-m_{01i}dP)y}dy\\
&=\sum_{d=1}^{D_0}\frac{2}{D_0}J(N,m_{01}dP)-F(P1(Py),m_{01},N)
\end{aligned}
$$

(17)

Replace the Euler function with following function

$$P1(x,m_{01})=\frac{1}{\pi}\left\{\sum_{\substack{n=1\\n\neq m_{01}}}^{\infty}\frac{\mathrm{Sin}(2n\pi x)}{n}+\frac{e^{2\pi i m_{01}x}}{2im_{01}}\right\}$$

in (17),thus could get the coefficient $C_0(\tau_0)$.we have

$$F(P1(Py),m_{01},N)=-C_0(\tau_0)J_0(N)+F(P1(Py,m_{01}),m_{01},N)$$

(18)

Next we change $\int_0^1\Rightarrow\int_{-\frac{1}{\tau}}^{1-\frac{1}{\tau}}$ in $S_1\oplus F(P1(Py,m_{01}),m_{01},N)$, Write

$$P1(x,m_{01i})=\bar{P}1(x,m_{01i})+\sum_{n>\tau^{\frac{2}{3}}}(\ )\text{'}$$

integration by parts and by Schwarz inequality, $\sum\limits_{n>\tau^{\frac{2}{3}}}^{\infty}$ in (18) is

$$\prec\prec \tau_0\tau_1 \frac{1}{N_3\tau^{\frac{2}{3}}}\frac{N^2}{\log N} \prec\prec \sqrt{N}$$

Thus

$$\begin{aligned} &F(P1(Py,m_{01}),m_{01},N) \\ &=\sum_{d=1}^{D_0}\int_{-\frac{1}{\tau}}^{1-\frac{1}{\tau}}\left\{\bar{P}1(dPy-dPE(dP),m_{01})-\bar{P}1(dPy+dPE(dP),m_{01})\right\}S^2(y)e^{-2i\pi(N-m_{01}dP)y}dy+O(\sqrt{N}) \\ &=F\left(\left(\bar{P}1(Py,m_{01}),m_{01},N\right),I_i(\tau_0)\right)+F\left(\left(\bar{P}1(Py,m_{01}),m_{01},N\right),I_2(\tau_0)\right)+O(\sqrt{N}) \end{aligned}$$

(19)

On the set $I_2(\tau_0)$,we are listed in the following formula:

$$\frac{1}{M_1N_3\prod\limits_{m=1}^{M_0}(N_3)^{\frac{1`}{2^m}}\left((D_0)^{\sum\limits_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)}\sum_{\substack{n=1\\n\neq m_{01}}}^{\tau^{\frac{2}{3}}}\sum_{m_{01}=M_1+1}^{2M_1}\prod_{m=1}^{M_0}\sum_{(N_3)^{\frac{1`}{2^m}}\left((D_0)^{\sum\limits_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)<P_m\le 2(N_3)^{\frac{1`}{2^m}}\left((D_0)^{\sum\limits_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)}\sum_{N_3<P_0\le 2N_3}\sum_{d=1}^{D_0}e^{-2\pi i(n-m_{01})dP\left(\frac{a}{q}+\frac{\theta}{q^2}\right)}\frac{Sin(2n\pi dPE(dP))}{2in\pi},\ P=\sum_{m=0}^{M_0}P_m$$

If $(n-m_{01})d<(D_0)^4$, sum P_0, then fixd n sum $(n-m_{01})d$,by lemma 1. this gives

$$\begin{aligned} &S_1\oplus F\left(\left(\bar{P}1(Py,m_{01}),m_{01},N\right),I_2(\tau_0)\right) \\ &\prec\prec \frac{(D_0)^{\varepsilon}\log\tau}{N_3M_1}\left(\left(\frac{(D_0)^4}{q}+1\right)q\log q\right)\int_{I_2(\tau_0)}\left|S^2(y)\right|dy \\ &\prec\prec \frac{N}{\tau_0\tau_1} \end{aligned}$$

(20)

If $(n-m_{01})d\ge(D_0)^4$,first sum P_0 ,then fixed m_{01} , d ,sum $(n-m_{01})d$ by lemma 1(logarithmic divide n into segments $\prec\prec \log\tau$), thus also gives

$$\begin{aligned} &S_1\oplus F\left(\left(\bar{P}1(Py,m_{01}),m_{01},N\right),I_2(\tau_0)\right) \\ &\prec\prec \frac{D_0}{N_3}\sum_{j=\left[\frac{3\log D_0}{\log 2}\right]}^{\log\tau}\frac{1}{2^j}\left(\left(\frac{\left|2^{J+1}+2M_1\right|D_0}{q}+1\right)(N_3+q\log q)\right)\int_{I_2(\tau_0)}\left|S^2(y)\right|dy \\ &\prec\prec \frac{N}{\tau_0\tau_1} \end{aligned}$$

(21)

It easy to see

$$\begin{aligned} &S_1\oplus\sum_{d=1}^{D_0}\int_{I_1(\tau_0)}\left\{\bar{P}1(dPy-dPE(dP),m_{01i})-\bar{P}1(dPy+dPE(dP),m_{01})\right\}S^2(y)e^{-2i\pi(N-m_{01}dP)y}dy \\ &=S_1\oplus\sum_{d=1}^{D_0}\int_{I_1(\tau_0)}\{P1(dPy-dPE(dP))-P1(dPy+dPE(dP))\}S^2(y)e^{-2i\pi(N-m_{01}dP)y}dy \\ &+S_1\oplus C_0(\tau_0)\int_{I_1(\tau_0)}S^2(y)e^{-2i\pi Ny}dy+O(\sqrt{N}) \end{aligned}$$

(22)

We use same method to calculate the function $F2((P,m_{01},N))$,we have

$$
\begin{aligned}
&S_1 \oplus F2((P, m_{01i}, N)) = S_1 \oplus C_1(\tau_0)(J_0(N) - J_0(N, I_1(\tau_0))) + \\
&+ S_1 \oplus \left(\sum_{d=2}^{\frac{D_0}{2}} \sum_{\substack{j=1 \\ (d,(D_0-d+j))=1}}^{d} + \sum_{d=\frac{D_0}{2}+1}^{D_0} \sum_{j=2d-D_0}^{d} \right) \left(\frac{j}{d(D_0-d+j)D_0} \right) \{ (J_0(N, dm_{01}P) + J_0(N, (D_0-d+j)m_{01}P)) \} \\
&- S_1 \oplus F2((P1(Py), m_{01}, N), I_1(\tau_0)) + O\left(\frac{N}{\sqrt{\tau_0 \tau_1}} \right)
\end{aligned}
\tag{23}
$$

We noticed that the functions on the set are all functions with period of 1,thus

$$
S_1 \oplus F((P1(Py), m_{01}, N), I_1(\tau_0)) = S_1 \oplus F((P, m_{01}, N), I_1^*(\tau_0)) + S_1 \oplus \sum_{d=1}^{D_0} \frac{2}{D_0} J_0(N, m_{01}dP, I_1(\tau_0))
$$

$$
,\quad I_1^*(\tau_0) = \begin{cases} I_1(\tau_0) \quad 1 < q < Q_1 \,,\ y = \dfrac{a}{q} + z \in I_1(\tau_0) \\ a = 0 \ or\ 1 \quad q = 1 \end{cases}
$$

$$
\begin{aligned}
&S_1 \oplus F2((P1(Py), m_{01}, N), I_1(\tau_0)) = S_1 \oplus F2((P, m_{01}, N), I_1^*(\tau_0)) \\
&+ S_1 \oplus \left(\sum_{d=2}^{\frac{D_0}{2}} \sum_{\substack{j=1 \\ (d,(D_0-d+j))=1}}^{d} + \sum_{d=\frac{D_0}{2}+1}^{D_0} \sum_{j=2d-D_0}^{d} \right) \left(\frac{j}{d(D_0-d+j)D_0} \right) \{ (J_0(N, dm_{01}P, I_1(\tau_0)) + J_0(N, (D_0-d+j)m_{01}P, I_1(\tau_0))) \}
\end{aligned}
$$

And

$$
S_1 \oplus F1((d_P P)_y, m_{01}, N, U(P) \setminus I_2(\tau_0)) + S_1 \oplus F2((P, m_{01}, N), I_1^*(\tau_0)) = S_1 \oplus F((P, m_{01}, N), I_1^*(\tau_0))
\tag{24}
$$

It easy to know

$$
S_1 \oplus \sum_{d=1}^{D_0} \frac{2}{D_0} J_0(N, m_{01}dP, I_2(\tau_0)) \prec\prec \frac{N}{M_1} \,, \quad etc.
\tag{25}
$$

By (15)~(25) the lemma 3 thus proved.

Theorem 1 proof. Let

$y = \frac{a}{q} + z_1 \in I_2(\tau_0)$,

$$
S_1 \oplus = \frac{1}{N_3 M_1 \prod_{m=1}^{M_0} (N_3)^{\frac{1`}{2^m}} \left( (D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right)} \sum_{m_{01}=M_1+}^{2M_1} \prod_{m=1}^{M_0} \sum_{(N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right) < P_m \le 2(N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right)} \sum_{N_3 < P_0 \le 2N_3} \quad if \quad \sqrt{N_3}(D_0)^3 < q \le \tau \ ,
$$

$$
S_1 \oplus = \frac{1}{N_3 M_1 \prod_{m=1}^{M_0} (N_3)^{\frac{1`}{2^m}} \left( (D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right)} \sum_{m_{01}=M_1+}^{2M_1} \sum_{N_3 < P_0 \le 2N_3} \prod_{\substack{a=1 \\ a \ne m}}^{M_0} \sum_{(N_3)^{\frac{1`}{2^a}} \left((D_0)^{\sum_{j=0}^{a-1} \frac{3}{2^j} + \frac{1}{2}} \right) < P_a \le 2(N_3)^{\frac{1`}{2^a}} \left( (D_0)^{\sum_{j=0}^{a-1} \frac{3}{2^j} + \frac{1}{2}} \right)} \sum_{(N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right) < P_m \le 2(N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right)} ,
$$

$$
if \quad (N_3)^{\frac{1`}{2^{m+1}}} \left( (D_0)^{\sum_{j=0}^{m} \frac{3}{2^j}} \right) < q \le (N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{mi-1} \frac{3}{2^j}} \right)
$$

If $(N_3)^{\frac{1`}{2^{m+1}}} \left( (D_0)^{\sum_{j=0}^{m} \frac{3}{2^j}} \right) < q \le (N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{mi-1} \frac{3}{2^j}} \right)$,let $q^{(m)} = q$ and

$$
P^{(m)} = P^{(m)}(P_0, \ldots, P_{m-1}, P_{m+1}, \ldots, P_{M_0}) = \sum_{\substack{a=1 \\ a \ne m}}^{M_0} P_a + P_m \,,\ (N_3)^{\frac{1`}{2^m}} \left( (D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right) < P_m \le 2(N_3)^{\frac{1`}{2^m}} \left((D_0)^{\sum_{j=0}^{m-1} \frac{3}{2^j} + \frac{1}{2}} \right)
$$

For by lemma 2,let set $I_2(\tau_0)$ contract with the following sets

$$
y = \frac{a}{q^{(m)}} + z_1 \in I_2(\tau_0) \,,\ \frac{a}{q^{(m)}} + z_1 = \frac{x}{d_{P^{(m)}} P^{(m)}} + z \,,\ 1 \le d_{P^{(m)}} \le D_0 \,,\ |z| \le \frac{1}{d_{P^{(m)}} P^{(m)} D_0}
\tag{A}
$$

Assume the equation $\frac{ad}{r} P^{(m)} - \frac{q^{(m)}}{r} x = 1$, $r = (d, q^{(m)})$ has solution $(P_{0d},\ x_0)$, the condition of interval intersection (A) is given from the following equation

$$adP^{(m)}-xq^{(m)}=lr\ ,\ r=\left(d,q^{(m)}\right)\quad l\le\frac{2q}{rD_0}\ ,$$

Thus gives

$$P_d^{(m)}=lP_{0d}+t\frac{q^{(m)}}{r},\frac{\left((N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)+P^{(m)}-P_m-lP_{0d}\right)r}{q}<t\le\frac{\left(2(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)+P^{(m)}-P_m-lP_{0d}\right)r}{q},\left(P_{0d},\frac{q^{(m)}}{r}\right)=1,1\le d\le D_0,1\le l\le\frac{2q^{(m)}}{rD_0}$$

,if $dP_d^{(m)}\equiv d_1P_{d_1}^{(m)}\ \left(q^{(m)}\right)$,then $d\left(l_1P_{0d}+t_1\frac{q^{(m)}}{r}\right)\equiv d_1\left(l_2P_{0d_1}+t_2\frac{q^{(m)}}{r}\right)\ \left(q^{(m)}\right)$,that is $adl_1P_{0d}\equiv ad_1l_2P_{0d_1}\ \left(q^{(m)}\right)$, for $adP_{0d}\equiv r\ \left(q^{(m)}\right)$,, $ad_1P_{0d_1}\equiv r_1\ \left(q^{(m)}\right)$, thus gives $rl_1\equiv r_1l_2\ \left(q^{(m)}\right)$, if $r=r_1\ \ l_1=l_2$, then

$$P_d^{(m)}P_{0d_1}-P_{d_1}^{(m)}P_{0d}=\left(t_1P_{0d}-t_2P_{0d_1}\right)\frac{q^{(m)}}{r}$$

For must exist $N_3<P_{0d},P_{0d_1}<3N_3$ satisfy (A),because the number of different prime factor of P_{0d_1} great than $\sqrt{D_0}$ is less than $\sqrt{\log N}$,for $\left(P_{0d},\frac{q^{(m)}}{r}\right)=\left(P_{0d_1},\frac{q^{(m)}}{r}\right)=1$,thus must exist $0\le i\le\left\lfloor\sqrt{\log N}\right\rfloor$, $P_{0d}^{/}=P_{0d}+(t_0+i)\frac{q^{(m)}}{r}$ makes $\left(P_{0d}^{/},P_{0d_1}\right)=r_3\le\sqrt{D_0}$,thus equation $P_d^{(m)}P_{0d_1}-P_{d_1}^{(m)}P_{0d}^{/}=\left(t_1P_{0d}^{/}-t_2P_{0d_1}\right)\frac{q^{(m)}}{r}$,only have

$$\prec\prec\sqrt{D_0}\left(\frac{r(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)}{q^{(m)}}\right)^2$$

values $P_d^{(m)},P_{d_1}^{(m)}$. If $r\ne r_1\ ,rl_1=r_1\,l_2$,then $(l_1,l_2)\ge\frac{\sqrt{q^{(m)}}}{D_0},l_1,l_2>\sqrt{q^{(m)}}$ .Thus by lemma 1 we have

$$\begin{aligned}
&S_1\oplus F1\left((d_PP)_y,m_{01},N,U(P)\cap I_2(\tau_0)\right)\\
&\prec\prec\max_{y,m}\frac{1}{(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)M_1}\left(\frac{(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)}{q^{(m)}}+1\right)\sum_{1\le l\le q^{(m)}}\left|\sum_{m_{91}>M_1}^{2M_1}e^{2\pi i m_{01}ly}\right|\int_{I_2(\tau_0)}\left|S^2(y)\right|dy+O\left(\frac{N}{D_0}\right)+O\left(\frac{(D_0)^2N}{\sqrt{Q_1}}\right)+O\left(\frac{M_1N_3D_0N}{Q_1\tau}\right)\\
&\prec\prec\max_{m}\frac{1}{(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)M_1}\left(\frac{(N_3)^{\frac{1}{2^m}}\left((D_0)^{\sum_{j=0}^{m-1}\frac{3}{2^j}+\frac{1}{2}}\right)}{q^{(m)}}+1\right)\sum_{1\le l\le q^{(m)}}\min\left(M_1,\frac{1}{\left\langle\left(\frac{a}{q^{(m)}}+\frac{\theta}{q^{(m)}\tau}\right)l\right\rangle}\right)\int_{I_2(\tau_0)}\left|S^2(y)\right|dy+O\left(\frac{N}{D_0}\right)\\
&\prec\prec\ \frac{N}{M_1}
\end{aligned}\tag{26}$$

Next we calculate the value of the coefficient

$$\begin{aligned}
S_1\oplus(C_0(\tau_0)-C_1(\tau_0))&=2-2\left(\sum_{d=2}^{\frac{D_0}{2}}\sum_{\substack{j=1\\(d,(D_0-d+j))=1}}^{d}+\sum_{d=\frac{D_0}{2}+1}^{D_0}\sum_{j=2d-D_0}^{d}\right)\frac{j}{d(D_0-d+j)D_0}+O\left(\frac{1}{\log^{b_2}N}\right)\\
&\ge2-2\sum_{d=2}^{\frac{D_0}{2}}\left(\frac{1}{dD_0}(d-(D_0-d)(\log D_0-\log((D_0-d+1))))\right)+\sum_{d=\frac{D_0}{2}+1}^{D_0}\left(\frac{1}{dD_0}((D_0-d)-(D_0-d)(\log D_0-\log d))\right)+O\left(\frac{1}{\log^{b_2}N}\right)\\
&=2-2\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right)\left(\frac{1}{2}\right)^{n+1}+\frac{1}{2}\log2-\frac{1}{2}(\log2)^2\right\}+O\left(\frac{1}{\log^{b_2}N}\right)\\
&=1.718+O\left(\frac{1}{\log^{b_2}N}\right)
\end{aligned}\tag{27}$$

By (26),(27) and lemma 3,theorem 1 thus proved.

Note,the theorem 1 also holds for if let $N=N+3-p_i\ ,2<p_i\le N+3$ be prime number.The theorem of three prime numbers tells us that $\tau<N$.

In order to prove theorem 2,lemma 4 and lemma 5 is proved first.

Lemma 4.Let

$$Q_i = e^{a_0^i (\log N)^{\frac{7}{12}}},\ a_o = \frac{6}{5},\ 0 \le i \le M_0,\ M_0 = \left[(\log a_0)^{-1} \log\left(\frac{2(\log N)^{\frac{5}{12}}}{(\log\log N)} \right) \right]$$

N be a sufficiently large integer, c_1 be a arbitrary positive number,then

$$\sum_{Q_0 < q \le Q_{M_0}} \left| \frac{\mu^2(q)}{\Phi^2(q)} C_q(-N) \right| \prec\prec \frac{1}{(Q_0)^{\frac{1}{7} - \varepsilon_2}}$$

Where ε_1 , ε_2 is a sufficiently small positive number.

Proof. We have

$$\sum_{Q_i \le q < Q_{i+1}} \left| \frac{\mu^2(q)}{\Phi^2(q)} C_q(-N) \right| = \sum_{d|N} \Phi(d) \sum_{\substack{Q_i \le q < Q_{i+1} \\ (q,N)=d}} \frac{\mu^2(q)}{\Phi^2(q)}$$

$$\prec\prec (\log\log Q)^2 \sum_{d|N} \frac{\mu^2(d)}{\Phi(d)} \sum_{Q_i/d \le v < Q_{i+1}/d} \frac{1}{v^2}$$

$$\prec\prec (Q_i)^{-1} (\log\log Q)^2 \sum_{\substack{d|N \\ d<Q_{i+1}}} \frac{\mu^2(d) d}{\Phi(d)}$$

Let $\nu_1(N)$ be the numbers of different prime factors of N, $\nu_1([Q_{i+1}]) = \min\left(\max_{q \le Q_{i+1}} \nu_1(q), \nu_1(N) \right)$,then

$$\sum_{\substack{d|N \\ d<Q_{i+1}}} \frac{\mu^2(d) d}{\Phi(d)} \prec\prec (\log\log N)\left(1 + C_{1`}^{\nu_1(N)} + \ldots + C_{\nu_1([Q_{i+1}])`}^{\nu_1(N)}\right) \prec\prec (\log N) C_{\nu([Q_{i+1}])}^{\nu(N)},$$

$$\nu_1(N) \le \nu(N) = \left[\frac{4\log N}{(\log\log N)} \right] \ and\ \ \nu_1([Q_{i+1}]) \le \nu([Q_{i+1}]) = \left[\frac{a_0^{i+1+\varepsilon} c_1 (\log N)^{\frac{13}{24}}}{\log\left(a_0^{i+1} c_1 (\log N)^{\frac{13}{24}} \right)} \right]$$

Hence

$$\sum_{\substack{d|N \\ d<Q_{i+1}}} \frac{\mu^2(d) d}{\Phi(d)} \prec\prec (\log N) \prod_{j=0}^{\left[\frac{a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}}{\log\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)} \right] - 1} \left(\frac{4\log N}{(\log\log N)} \right) \Bigg/ \prod_{j=0}^{\left[\frac{a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}}{\log\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)} \right] - 1} \frac{a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}}{\log\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)} \left(1 - \frac{j}{\frac{a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}}{\log\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)}} \right)$$

$$\prec\prec \log N e^{\frac{a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}}{\log\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)} \left(\frac{5}{12} (\log\log N) + \log \frac{4}{a_0^{i+1}} \right)} \frac{e^{-\frac{a_0^{2+2i} (\log N)^{\frac{1}{6}}}{8\log^2\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)} \log\log N}}{e^{-\frac{a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}}{\log\left(a_0^{i+1} (\log N)^{\frac{7}{12}} \right)}}} \prec\prec e^{\frac{5}{7} a_0^{i+1+\varepsilon} (\log N)^{\frac{7}{12}}} = (Q_i)^{\frac{6}{7} + \varepsilon_1}$$

Thus

$$\sum_{Q_0 \le q < Q_{M_0}} \left| \frac{\mu^2(q)}{\Phi^2(q)} C_q(-N) \right| \prec\prec \sum_{0 \le i \le M_0} \frac{1}{(Q_i)^{\frac{1}{7} - \varepsilon_1}} \prec\prec \frac{1}{(Q_0)^{\frac{1}{7} - \varepsilon_2}}$$

This proves the lemma 4.

Lemma 5. Given the parameters in lemma 3, Let

$$S_2 \oplus FG((P1(py), m_{01}, N), I_1(\tau_0)) = S_2 \oplus \tau_0 \tau_1 \int_{I_1(\tau_0)} \{P1(Py + PE_1(2P)) - P1(Py + PE_1(P))\} S^2(y) e^{-2i\pi y(N - m_{01}P)} dz$$

$$= S_2 \oplus FG((P, m_{01}, N), I_1(\tau_0)) + \frac{1}{4} S_2 \oplus J_0(N, m_{01}P, I_1(\tau_0)) \quad , \quad E_1(P) = \frac{1}{2\tau_0 \tau_1 P},_3$$

Then

$$S_2 \oplus FG((P1(py), m_{01}, N), I_9(P, \tau_0)) = S_2 \oplus \frac{1}{4} J_0(N, m_{01}P, I_1(\tau_0)) + S_2 \oplus \tau_0\tau_1 \int_{\tilde{I}_3(\tau_0)} s^2(y) e^{-2i\pi y(N - m_{01}P)} dy + \Omega D_{1,4}(N, \tau_0) + O\left(\frac{N}{e^{\frac{c_{10}\log N}{\log Q_1}}}\right) + O\left(\frac{N}{(\tau_0\tau_1)^{\frac{1}{8}}}\right)$$

Where

$$S_2 \oplus = \sum_{m_{01}=M_1+1}^{2M_1} \sum_{P=N_3+1}^{2N_3} \frac{1}{M_1 N_3}$$

$$\Omega D_{1,4}(N, \tau_0) \prec\prec \log N \left| \sum_{\frac{\tau_0\tau_1}{q_0} < k \le \frac{Q_i^2}{q_0}} \tilde{\chi}^2(k) \frac{\mu^2(k) C_k(-N)}{\phi^2(k)} \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} \left(\tilde{T}(z)\right)^2 e(-Nz) dz \right|$$

$$\tilde{I}_3(\tau_0) \equiv \frac{a}{q} + z \in I_1(\tau_0), \quad -E_1(P) \le z \le -E_1(2P)$$

Proof.Let $I_3(p) \equiv \frac{x}{P} + z$, $x = 0, \ldots, P-1$, $-E_1(P) \le z \le -E_1(2P)$,we analyse the value on the $I_3(P) \cap I_1(\tau_0)$,

If $\frac{a}{q} \ne \frac{x}{P}$ let $I_4(\tau_0) = I_3(\tau_0) \cap I_5$ in which $I_5 \in I_1(\tau_0)$: $q < \tau_0\tau_1$, $|z| \le \frac{1}{2qP}$. ,For $\left|\frac{a}{q} - \frac{x}{P}\right| = \frac{|P - qx|}{qP} \ge \frac{1}{2qP} + E_1(P)$,hence $I_4(\tau_0) = \Phi$ (null set),let $I_6 = I_1(\tau_0) \setminus I_5$ and $I_{6A}(\tau_0) \in I_6$ in which $q < \tau_0\tau_1$, $\frac{1}{2qP} < |z| \le \frac{1}{q(\tau_0\tau_1)\tau}$, For here

$y \in I_{6A}(\tau_0) \cap I_3(\tau_0)$, $y = \frac{h_1}{P_1} + z$, where

$$P_1 \ge \frac{1}{q} / \left(\frac{1}{q(\tau_0\tau_1)\tau} + \frac{1}{2\tau_0\tau_1 P}\right)^{-1} > \frac{\tau}{Q_1},$$

Sum m_{01} by Abel transform then sum P by Lemma 1,(3)~(12) gives

$$\begin{aligned} & S_1 \oplus FG((P1(py), m_{01}, N), I_{6A}(\tau_0)) \\ & \prec\prec \frac{\tau_0\tau_1}{M_1 N_3} \sum_P \min\left(M_1, \frac{1}{\left\langle \frac{h_1}{P_1} * P \right\rangle}\right) \int_{i_{6A}(\tau_0)} |S^2(y)| dy \\ & \prec\prec (\log N)^{b_2+2} \int_{i_{6A}(\tau_0)} |S^2(y)| dy \\ & \prec\prec \log^{b_2+2} N \left(\frac{\tau_0\tau_1 N_3}{\log N} + \frac{N}{e^{\frac{c_{10}\log N}{\log(\tau_0\tau_1)}}}\right) \\ & \prec\prec \frac{N}{\tau_0} + \frac{N}{e^{\frac{c_{10}\log N}{\log(\tau_0\tau_1)}}} \end{aligned} \tag{28}$$

2). IF $\frac{a}{q} = \frac{x}{P}$,use $\tilde{I}_3(\tau_0)$ to represent this set.

3). $q \nmid P$,. define set $I_8(\tau_0) \in I_1(\tau_0)$, $\tau_0\tau_1 \le q < Q_1$, $I_9(P, \tau_0) = I_3(\tau_0) \cap I_8(\tau_0)$.we analysis the value on it. Let $r = (q, P_{01}P)$,then $\frac{a}{q} - \frac{x}{P} = \frac{aP - xq}{qP} = \frac{n_{01}r}{qP}$,this gives the number of intersection interval,namely for fixed q and P , the numbers of the pairs (a, x) of equation

$$aP - xq = n_{01}r, \ (n_{01}, q) = 1, \quad n_{01} \le \frac{I(q, P)}{r},$$

and the condition of interval intersection is given

$$I(q, P) < 2\left(\frac{qP}{q(\tau_0\tau_1)\tau} + \frac{q}{2\tau_0\tau_1}\right) \tag{29}$$

For $\left(\frac{q}{r}, \frac{P}{r}\right) = 1$,the equation $\frac{P}{\mathrm{r}} y - \frac{q}{r} x = 1$ has solution $(y_0, \ x_0)$.Hence the equation $Pa - qx = r$,$(a \ \ x)$ just has solution $\left(y_0 + t\frac{q}{r}, x_0 + t\frac{P}{r}\right)$, $t = 0, 1, \ldots, r-1$,but here

$\left(\frac{q}{r},r\right)=1$,there exists $t_{i,}\ 0<i\le\varphi(r)$ makes $\left(y_0+t_i\frac{q}{r},r\right)=1$,thus the complete intersecting set could defined as following

$$\frac{a_l^{(1)}}{q}\pm\frac{r}{qP(k,l)}\pm z,...,\frac{a_l^{(\phi(r))}}{q}\pm\frac{r}{qP(k,l)}+z,...,\frac{n_{01}a_l^{(t)}}{q}\pm\frac{n_{01}r}{qP(k,l)}+z,...,\frac{n_{01}a_l^{(\phi(r))}}{q}\pm\frac{n_{01}r}{qP(k,l)}+z$$
$$,...,\frac{R(k,l)a_l^{(1)}}{q}\pm\frac{R(k,l)r}{qP(k,l)}+z,...,\frac{R(k,l)a_l^{(\phi(r))}}{q}\pm\frac{R(k,l)r}{qP(k,l)}+z$$

Where

$$1\le n_{01}\le R(k,l)\le\frac{I(q,P(k,l))}{r},(n_{01},q)=1,P(k,l)=kq+rl,l<\frac{q}{r},\left(l,\frac{q}{r}\right)=1,-E(P)\le z\le -E(2P),$$

$P(k,l)a_l^t-qx=r\Rightarrow a_l^{(t)}l\equiv 1\ (q)$, $\{n_{01}a_l^{(t)}\},t\le\varphi(r),l\le\varphi\left(\frac{q}{r}\right)$ just go through the reduced residual of module q.

(30)

The series expansion will be the following form,note $a_l(i)*(rl)\equiv r\ (q)$,

$$e^{2\pi i m_{01}(kq+rl)\left(\frac{n_{01}a_l^{(t)}}{q}+z\right)}=e^{2\pi i(m_{01}rl)\left(\frac{n_{01}a_l^{(t)}}{q}\right)}e^{2\pi i m_{01}kqz}(1+2\pi i m_{01}rlz+...)$$
$$=e^{2\pi i\left(\frac{m_{01}n_{01}r}{q}\right)}e^{2\pi i m_{01}kqz}+O\left(\frac{M_1Q_1}{\tau_0\tau_1\tau}\right)$$

(31)

Finally by(29)~(31) ,we have the value on $I_9(P,\tau_0)$

$$S_2\oplus FG((P,m_{01},N),I_9(P,\tau_0))$$
$$\le\frac{\tau_0\tau_1}{N_3M_1}\sum_{\tau_0\tau_1\le q<Q_1}\sum_{r|q}\sum_{k=0}^{\left[\frac{N_3}{q}\right]-1}\sum_{1\le n_{01}\le\frac{I(q,P(k.1))}{r}}\sum_{M_1<m_{01}\le 2M_1}e^{2\pi i\left(\frac{m_{01}n_{01}r}{q}+2\pi i m_{01}(N_3+kq)z\right)}\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\left|\sum_{t\le\varphi(r)}\sum_{l\le\varphi\left(\frac{q}{r}\right)}S^2\left(\frac{n_{01}a_l^{(t)}}{q}+z\right)e^{-2\pi i\left(\frac{n_{01}a_l^{(t)}}{q}+z\right)N}\right|dz$$
$$+\frac{M_1Q_1}{N_3\tau}\sum_{\tau_0\tau_1\le q<Q_1}\sum_{k=0}^{\left[\frac{N_3}{q}\right]-1}\sum_{1\le n\le I(q,P(k.1))}d(n)\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{q(\tau_0\tau_1)(N_5)}}^{\frac{1}{q(\tau_0\tau_1)(N_{35})}}\left|S^2\left(\frac{a}{q}+z\right)\right|dz$$

(3)~(12) and lemma 4 gyves the value

$$S_2\oplus FG((P,m_{01},N),I_9(P,\tau_0))=S_2\oplus\tau_0\tau_1\int_{\tilde{I}_3(\tau_0)}s^2(y)e^{-2i\pi y(N-m_{01}P)}dy+\Omega D_{1,4}(N,\tau_0)+O\left(\frac{N}{(\tau_0\tau_1)^{\frac{1}{7}-\varepsilon_2}}\right)+O\left(\frac{N\log^2 N}{e^{\frac{c_{10}\log N}{\log Q_1}}}\right)$$

(32)

By (28),(32),the proof of lemma 5 is thus completed.

Let q_0 be a exception module(chapter 10 lemma7 of reference 4)

Theorem 2.(Real Zero Distribution of Exception Module **)**

Let A_1 be a sufficiently large positive number, $\log^{A_1}N<q_0\le Q_1$, $\tilde{\chi}$ be a real primary feature module q_0, β_0 be a real zero of $L(s,\chi)$, $\chi\bmod q\ \chi\Leftrightarrow\tilde{\chi}\quad q_0\mid q$. when N is fully large then

$$\beta_0\le\max\left(1-\frac{\log q_0}{25\log N},1-\frac{c_{10}}{21\log Q_1}\right)$$

Where c_{10} be a constant in (11).

proof. Corresponding to lemma 3 and lemma 5, Redefine the following parameters

$$\tau_1=\min\left((q_0)^{\frac{1}{12}},e^{\frac{c_{10}}{10}\frac{\log N}{\log Q_1}}\right),\tau_0=q_0(\tau_1)^8,N_3=\left[\frac{N}{(Q_1)^3}\right]$$
$$S_1^*\oplus=\sum_{\substack{M_{0t}=m_{01}+...+m_{0t}\\M_1+1<m_{01}\le 2M_1}}\sum_{N_3<P\le 2N_3}\frac{1}{N_3(M_1)^t},t=\left[\frac{\log N}{\log\log N}\right]$$

Define an N^* related q_0 instead of N ,as we know $q_0 = 2^a p_1 ... p_s$, $a = 0,2,3$,we define

$$N^* = 2^{n_{03}} p_1 ... p_s \ , \ n_{03} = \left[\frac{1}{\log 2} \log\left(\frac{N}{p_1 ... p_s} \right) \right] + 1$$

hence $N^* = 2^{\vartheta} N$, $0 < \vartheta \le 1$ and $q_0 \mid N^*$

Let E_{q_0} be a set, if $y \in E_{q_0}$,then

$$y = \frac{a}{q} + z, 0 \le a < q, (a,q) = 1, q_0 \le q < Q_1 \ , q_0 \mid q \ , |z| \le \frac{1}{q(\tau_0 \tau_1)\tau} \ , \tau = 10 N_3$$

Consider following equation

$$S_1^* \oplus \sum_{w=1}^{2} F^*\left(\left(\left(P1(Py, M_{0t})\right), M_{0t}, N^*\right), E_{q_0}\right)$$

$$= S_1^* \oplus \sum_{w=1}^{2} F^*\left(\left(P1(Py), M_{0t}, N^*\right), E_{q_0}\right) + S_1 \oplus C_o^*(\tau_0) \sum_{\substack{q_0 < q \le Q_1 \\ q_0 \mid q}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} S^2\left(\frac{a}{q} + z\right) e^{-2i\pi\left(\frac{a}{q}+z\right)N^*} dz$$

Where

$$S_1^* \oplus F^*\left(\left(\left(P1(Py)\right), M_{0t}, N^*\right), E_{q_0}\right) = S_1^* \oplus \tau_0 \tau_1 \int_{E_{q_0}} \{P1(Py + PE_1(2P)) - P1(Py + PE_1(P))\} S^2(y) e^{-2i\pi y(N^* - M_{0t}P)} dz$$

$$S_1^* \oplus C_o^*(\tau_0) = S_1^* \oplus \tau_0 \tau_1 \frac{e^{-2\pi i M_{0t} PE_1(2P)} - e^{-2\pi i M_{0t} PE_1(P)}}{2\pi i M_{0t}} \tag{33}$$

First let

$$P1(x, M_{0t}) = P3(x, M_{0t}) + \sum_{n > \tau^{\frac{2}{3}}}^{\infty} (\ )$$

$$S_1 \oplus F^*\left(\left(P1(P' y, M_{0t}), M_{0t}, N^*\right), E_{q_0}\right)$$

$$= \frac{\tau_0 \tau_1}{N_3 (M_1)^t} \sum_{\substack{M_1 < m_{0i} \le M_1 + 1 \\ M_{0t} = \sum_{i=1}^{t} m_{0i}}} \left\{ \sum_{\substack{n=1 \\ n \ne M_{0t}}}^{\tau^{\frac{2}{3}}} \sum_{P=N_3+1}^{2N_3} e^{-2\pi i (n - M_{0t}) P\left(\frac{a}{q} + \frac{\theta}{q^2}\right)} \frac{Sin(n\pi PE_1(2P)) e^{-3n\pi i PE_1(2P)}}{2n\pi} + \sum_{n=1}^{\tau^{\frac{2}{3}}} \sum_{P=N_3+1}^{2N_3} e^{-2\pi i (n + M_{0t}) P\left(\frac{a}{q} + \frac{\theta}{q^2}\right)} \frac{Sin(n\pi PE_1(2P)) e^{3n\pi i PE_1(2P)}}{2n\pi} \right\} + O\left(\sqrt{N}\right)$$

We calculate first one,sum P ,this gives unprotected assume $q \mid (n - M_{0t})$, $n = M_{0t} + kq$,if $n \le \frac{q}{\tau_1}$ then $q < M_1 \log N$,by lemma 1,

$$S_1^* \oplus F^*\left(\left(\left(P1(P' y, M_{0t})\right), M_{0t}, N^*\right), E_{q_0}\right)$$

$$\prec\prec \frac{q}{N_3 M_1 \tau_1} \left(\frac{M_1}{q} + 1 \right) \left(N_3 + q \log q^t \right) \frac{N}{\log N}$$

$$\prec\prec \frac{N}{(q_0)^{\frac{1}{12}}} + \frac{N}{e^{\frac{c_{10}}{10} \frac{\log N}{\log Q_1}}} \tag{34}$$

If $n > \frac{q}{\tau_1}$, $n = M_{0t} + kq$,then if $q \mid P$,thus

$$S_1^* \oplus F^*\left(\left(\left(P1(P' y, M_{0t})\right), M_{0t}, N^*\right), E_{q_0}\right)$$

$$\prec\prec \frac{\tau_0 \tau_1}{q} \sum_{\substack{k \\ n = |kq + M_{0t}| > \frac{q}{\tau_1}}} \frac{1}{|kq + M_{0t}|} \frac{N}{\log N}$$

$$\prec\prec \frac{(\tau_1)^{11} N}{q} \prec\prec \frac{N}{(q_0)^{\frac{1}{12}}} \tag{35}$$

if $q \nmid P$, $q \le \frac{M_1}{\log N}$,sum M_{0t} ,then sum P by lemma 1,we have

$$S_1^* \oplus F^*\left(\left(\left(P1(P' y, M_{0t})\right), M_{0t}, N^*\right), E_{q_0}\right)$$

$$\prec\prec \frac{\tau_0\tau_1 \log\tau}{N_3(M_1)^t}\left(\frac{N_3}{q}+1\right)\left(\frac{5}{4}q\right)^t \frac{N}{\log N}$$
$$\prec\prec \sqrt{N} \tag{36}$$

Otherwise $q > \frac{M_1}{\log N}$, $n = M_{0t} + kq \geq \frac{q}{\tau_1}$,(3)~(12) gives

$$S_{_1}^* \oplus F^*\left(\left(\left(P1\left(P^{/} y, M_{0t}\right)\right), M_{0t}, N^*\right), E_{q_0}\right)$$
$$\prec\prec S_{_1}^* \oplus \tau_0\tau_1 \sum_{\substack{k \\ \left[\frac{q}{\tau_1}\right] \leq n(k) \leq \tau^{\frac{2}{3}}}} \frac{1}{|kq+M_{0t}|} \sum_{\substack{\frac{M_1}{\log N} \leq q \leq Q_1 \\ q_0|q}} \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} \left| \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S^2\left(\frac{a}{q}+z\right) e^{-2i\pi\left(\frac{a}{q}+z\right)N^* - kqPz} \right| dz$$
$$\prec\prec \tau_0(\tau_1)^3 \frac{\log N}{M_1} \left\{ \sum_{\frac{M_1}{\log N} < q \leq Q_1} \left| \frac{\mu^2(q) C_q(-N^*)}{\phi^2(q)} \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} \left(\tilde{T}(z)\right)^2 e(-N^* z) dz \right| \right.$$
$$\left. + \sum_{\frac{M_1}{q_0 \log N} < k \leq \frac{Q_1}{q_0}} \left| \tilde{\chi}^2(k) \frac{\mu^2(k) C_k(-N^*)}{\phi^2(k)} \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} \left(\tilde{T}(z)\right)^2 e(-N^* z) dz \right| \right\} + O\left(\frac{N \log^2 N}{e^{\frac{7c_{10}}{10}\frac{\log N}{\log Q_1}}}\right)$$

If $\tau_1 = (q_0)^{\frac{1}{12}}$,for $2^{\nu_1(N^*)} \leq 2^{\nu_1(q_0)+1} \prec\prec (q_0)^{\varepsilon_1}$,then

$$S_{_1}^* \oplus F^*\left(\left(\left(P1\left(Py, M_{0t}\right)\right), M_{0t}, N^*\right), E_{q_0}\right)$$
$$\prec\prec \frac{N}{\sqrt{q_0}} + \frac{N}{e^{\frac{7c_{10}}{10}\frac{\log N}{\log Q_1}}} \tag{37}$$

If $\tau_1 = e^{\frac{c_{10}}{10}\frac{\log N}{\log Q_1}}$,we use the same proof method as lemma 4,could obtained

$$S_{_1}^* \oplus F^*\left(\left(\left(P1\left(Py, M_{0t}\right)\right), M_{0t}, N^*\right), E_{q_0}\right)$$
$$\prec\prec \left(\log^{b_2+2} N\right) e^{\frac{2c_{10}}{10}\frac{\log N}{\log Q_1}} \sum_{e^{\frac{9c_{10}}{10}\frac{\log N}{\log Q_1}-\varepsilon_1} < k \leq Q_1} \tilde{\chi}^2(k) \frac{\mu^2(k) C_k(-N^*)}{\phi^2(k)} \left| \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} \left(\tilde{T}(z)\right)^2 e(-N^* z) dz \right| + \frac{N}{e^{\frac{c_6}{10}\frac{\log N}{\log Q_1}}}$$
$$\prec\prec \frac{N}{e^{\frac{c_{10}}{5}\frac{\log N}{\log Q_1}}} + \frac{N}{e^{\frac{7c_{10}}{10}\frac{\log N}{\log Q_1}}} \tag{38}$$

On the other hand,by lemma 5,first we have

$$S_{_1}^* \oplus J_0\left(N, M_{0t}P^{/}, E_{q_0}\right) = \frac{1}{N_3(M_1)^t} \sum_{N_3 < P \leq 2N_3} \sum_{\substack{\sum \\ M_{0t}=\sum_{i=1}^{t} m_{0i} \\ M_1 < m_{0i} \leq 2M_1}} \sum_{\substack{q_0 < q \leq Q_1 \\ q_0|q}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}} S^2\left(\frac{a}{q}+z\right) e^{-2i\pi\left(\frac{a}{q}+z\right)\left(N^* - M_{0t}P\right)} dz$$
$$\prec\prec \frac{N}{q_0} \tag{39}$$

Secondly,

$$S_{_1}^* \oplus F^*\left(\left(P, M_{0t}, N^{*,}\right), \tilde{I}_3(\tau_0) \cap E_{q_0}\right)$$
$$= \tau_0\tau_1 \frac{1}{N_3(M_1)^t} \sum_{N_3 < P \leq 2N_3} \sum_{\substack{\sum \\ M_{0t}=\sum_{i=1}^{t} m_{0i} \\ M_1 < m_{0i} \leq 2M_1}} \sum_{\substack{\sum \\ M_{0t}=\sum_{i=1}^{t} m_{0i} \\ M_1 < m_{0i} \leq 2M_1}} \sum_{\substack{q_0 < q \leq Q_1 \\ q_0|q, q|P}} \frac{1}{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-E_1(2P)}^{-E_1(P)} S^2\left(\frac{a}{q}+z\right) e^{-2i\pi\left(\frac{a}{q}+z\right)\left(N^* - M_{0t}Pz\right)} dz$$
$$\prec\prec (\tau_1)^9 \left(\tau_0\tau_1 N_3 + \frac{N}{e^{\frac{c_{10}\log N}{\log Q_1}}}\right)$$
$$\prec\prec \left(\frac{N}{Q_1} + \frac{N}{e^{\frac{c_{10}}{10}\frac{\log N}{\log Q_1}}}\right) \tag{40}$$

Because according to the conditions of the hypothesis set,

$$\frac{a}{q}+z\bigcap I_3(P)=\Phi,\quad q<\tau_0\tau_1.$$

Thus by lemma 5,(39),(40),same as (37),(38),we have

$$S_1^*\oplus F^*((P1(Py),M_{0t},N^*),E_{q_0})$$
$$\prec\prec \log N\left|\sum_{\frac{\tau_0\tau_1}{q_0}\le k\le\frac{Q_1}{q_0}}\tilde{\chi}^2(k)\frac{\mu^2(k)C_k(-N^*)}{\phi^2(k)}\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\left(\tilde{T}(z)\right)^2 e(-N^*z)dz\right|+O\left(\frac{N}{q_0}\right)+O\left(\frac{N\log^2 N}{e^{\frac{c_{10}\log N}{10\log Q_1}}}\right)$$
$$\prec\prec O\left(\frac{N}{(q_0)^{\frac{2}{3}}}\right)+O\left(\frac{N\log^2 N}{e^{\frac{2c_{10}\log N}{5\log Q_1}}}\right) \tag{41}$$

Finally by (33)~(41), (3)~(12) could obtain

$$\sum_{\substack{q_0\le q\le Q_0\\ q_0|q}}\frac{\mu^2(q)}{\phi^2(q)}C_q(-N^*)\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\left(\sum_{N_2\le n\le N^*}\frac{e^{2\pi inz}}{\log n}\right)^2 e^{-2\pi izN^*}dz$$
$$+\tilde{\tilde{\chi}}(-1)\frac{q_0}{\phi(q_0)}\sum_{0<k\le\frac{Q_0}{q_0}}\tilde{\tilde{\chi}}^2(k)\frac{\mu^2(k)}{\phi^2(k)}C_k(-N^*)\int_{-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{1}{q(\tau_0\tau_1)\tau}}\left(\sum_{N_2\le n\le N^*}\frac{n^{\beta_0-1}e^{2\pi inz}}{\log n}\right)^2 e^{-2\pi izN^*}dz$$
$$=O\left(\frac{N^*}{(q_0)^{\frac{1}{12}}}\right)+O\left(\frac{N^*}{e^{\frac{c_{10}\log N^*}{10\log Q_0}}}\right)$$

That is

$$\frac{(N^*)^{2\beta_0-1}}{\log^2 N^*}<\mu^2(q_0)\frac{N^*}{\phi(q_0)\log^2 N^*}+O\left(\frac{N^*}{(q_0)^{\frac{1}{12}}}\right)+O\left(\frac{N^*}{e^{\frac{c_{10}\log N}{10\log Q_1}}}\right)$$

Namely

$$\beta_0\le\max\left(1-\frac{\log q_0}{25\log N},1-\frac{c_{10}}{21\log Q_1}\right)$$

Theorem 2 thus proved.

Theorem **3. (Asymptotic Formula of Solution Number)**

Let $D(N)$ be the number of solutions of $p_1+p_2=N$, where p_1,p_2 be prime numbers, N be a even integer,when N is fully large

$$D(N)=\Theta(N)\frac{N}{\log^2 N}+O\left(\frac{N\log\log N}{\log^3 N}\right)$$

where

$$\Theta(N)=\prod_{p\not|N}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\left(1+\frac{1}{(p-1)}\right),$$

proof. By the Schwarz inequality, $N_2=\frac{N}{\log^{2b_2}N}$,thus

$$D(N)=J_0(N)+O\left(\frac{N}{\log^{b_2+1}N}\right)$$

Theorem 1 thus gives

$$D(N)=\sum_{q\le Q_1,}\sum_{\substack{a=1\\(a,q)=1}}^{q_2}\int_{\frac{a}{q}-\frac{1}{q(\tau_0\tau_1)\tau}}^{\frac{a}{q}+\frac{1}{q(\tau_0\tau_1)\tau}}S^2(\alpha)e(-N\alpha)d\alpha+O((\log^{2b_2}N)\Omega D_{1,4}(N,\tau_0))+O\left(\frac{N}{\log^3 N}\right)$$

By lemma 4, $2^{\nu(N)} \leq 2^{\frac{2\log N}{\log\log N}}$, (3) ~(12) easy to show

$$D_{1,1}(N) = \Theta(N)\frac{N}{\log^2 N} + O\left(\frac{N\log\log N}{\log^3 N}\right)$$

In (4)~(12) still need to care $D_{14}(N), D_{15}(N)$, when $q_0 \leq \log^{A_1} N$,with the Sicgcl's theorem $\beta_0 \leq 1 - \frac{c(\varepsilon)}{\log^{\varepsilon A_1} N}$,take ε makes $\varepsilon A_1 < \frac{1}{10}$.if $\log^{A_1} N < q_0 \leq Q_1$ using theorem 2 . thus prove theorem 3.

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