

Is Spacetime Non-metric?

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Abstract: If one assumes higher dimensions and that dimensional reduction from higher dimensions produces scalar-tensor theory and also that Palatini variation is the correct method of varying scalar-tensor theory then spacetime is nonmetric. Palatini variation of Jordan frame lagrangians gives an equation relating the dilaton to the object of non-metricity and hence the existence of the dilaton implies that the spacetime connection is more general than that given solely by the Christoffel symbol of general relativity. Transferring from Jordan to Einstein frame, which connection, lagrangian, field equations and stress conservation equations occur are discussed: it is found that the Jordan frame has more information, this can be expressed in several ways, the simplest is that the extra information corresponds to the function multiplying the Ricci scalar in the action. The Einstein frame has the advantages that stress conservation implies no currents and that the field equations are easier to work with. This is illustrated by application to Robertson-Walker spacetime.

Keywords: Jordan frame; Einstein frame; non-metricity; palatini variation

1. Introduction

There are at least four motivations for studying the frame dependence of scalar-tensor theory. In order of ascending importance these are: frame equivalence, spacetime connection and geometric description motivations. The frame equivalence motivation is to find out if the various frames provide the same description of physics. Whether the Jordan or Einstein frame or both or neither is the physical one has been discussed in^[9,14,15]. There are various methods of evaluating the physical worth of the field equations corresponding to the various frames: firstly experiments and observations^[27,7], secondly principles, such as the principle of equivalence and so on, thirdly energy conditions, although in the case of energy conditions it can be thought of the other way round: the value of the energy conditions or otherwise is demonstrated by what they predict in various frames, fourthly possible applications to quantum theory^[1,2,3]. In general relativity the spacetime connection is the Christoffel connection and this is derived from Palatini variation of the Einstein-Hilbert action. The history of the Palatini variation is not straightforward, see^[16,11,12]. Usually Palatini variation is taken to give the connection, sometimes it is taken to give field equations^[13,10,8,26,6]. Applying the Palatini variation to the Jordan frame gives a non-metric connection. The only place where Palatini variation of the Jordan frame is looked at is in^[4,5]; however there it is not explicitly stated that the existence of a dilaton forces the geometry of spacetime to be a Weyl geometry, which was suggested on string theory grounds in^[23]. There are usually considered to be two types of frame: the Jordan frame and the Einstein frame; perhaps because of the geometry involved it is better to call the Jordan frame with non-metricity the Weyl frame. The geometric description motivation^[23] is that at all levels of description a geometric description is preferable to a matter field description. In other words it is preferable to have a description involving the geometric side of the field equations to the stress side. In the present case this means that it is preferable to have the dilaton as a geometric object rather than a matter field, and this indeed happens via equation (9),

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which shows that the dilaton can be thought of as a function of the scalar object of non-metricity.

The conventions used are signature $-+++$ and Riemann tensor

$$R^{\alpha}_{\beta\nu\delta} \equiv 2\partial_{[\gamma}\Gamma^{\alpha}_{\delta]R} + 2\Gamma^{\alpha}_{[\nu|\rho]}\Gamma^{\rho}_{\delta]R}. \quad (1)$$

Usually barred tensors are constructed with the Christoffel symbol, usually denotes covariant derivative with respect to the full connection. Greek indices range over all dimensions, latin indices range over the three spatial dimensions.

2. The Jordan Frame

The Jordan frame action is taken to be

$$S = \int d^d x \sqrt{-g} \{ \mathcal{A}(\Phi)R - \mathcal{B}(\Phi)(\nabla\Phi)^2 - V(\Phi) \} + S_m[\exp(2\alpha(\Phi))g_{\mu\nu}, \psi_m], \quad (2)$$

where here \mathcal{A} is called the primary dilaton function, \mathcal{B} is the secondary dilaton function and V is the dilaton potential, S_m is the matter action, compare [7,14]. $\mathcal{A}(\Phi)$ is written as \mathcal{A} when the ellipsis is clear. The action (2) depends on three fields: $\{g_{\mu\nu}, \Phi, \psi_m\}$ and has four freely specifiable functions of Φ : $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), V(\Phi), \alpha(\Phi)\}$.

Assuming a symmetric metric and a symmetric connection it is straightforward to perform a Palatini variation by varying the connection, compare [4] equation (8)

$$\frac{\delta^{\mu\nu}}{\sqrt{-g}} \left[-2(\sqrt{-g}g^{\mu\nu}\mathcal{A})_{;\lambda} + (\sqrt{-g}\mathcal{A}(\delta^{\mu\nu}_\lambda g^{\mu\rho} + \delta^{\mu\rho}_\lambda g^{\mu\nu}))_{;\nu} \right] = 0, \quad (3)$$

[5] has more detailed calculations. The derivative of the determinant obeys

$$(\sqrt{-g})_{;\lambda} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\mu\nu}_{;\lambda}, \quad (4)$$

using the λ, μ trace to remove the middle term and 4 to remove the last term in the μ, ν trace gives

$$(2-d)A\nabla_\lambda\sqrt{-g} = d\mathcal{A}'\Phi_\lambda\sqrt{-g}, \quad (5)$$

now use the λ, μ trace and 5 to remove $g^{\rho\nu}_{;\rho}$ and $(\sqrt{-g})_{;\lambda}$ from 3 to give

$$\nabla_\lambda g^{\mu\nu} = \frac{2\mathcal{A}'}{\mathcal{A}(2-d)}\Phi_\lambda g^{\mu\nu}. \quad (6)$$

The object of non-metricity is defined in terms of the covariant derivative applied to the covariant form of the metric, requiring that the metric has an inverse gives the contravariant form

$$Q^{\mu\nu}_{;\lambda} \equiv \nabla_\lambda g^{\mu\nu}, \quad g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}, \quad \nabla_\lambda g_{\mu\nu} = -Q_{\lambda\mu\nu}. \quad (7)$$

For a Weyl geometry the object of non-metricity reduces to

$$\nabla_\lambda g^{\mu\nu} = Q_\lambda g^{\mu\nu}, \quad (8)$$

in the present case using 6 the vector Q_λ is a gradient vector $\partial_\lambda Q$ with

$$Q = \frac{2}{\mathcal{A}(2-d)} \ln(\mathcal{A}), \quad (9)$$

which relates the object of non-metricity Q to the primary dilaton function \mathcal{A} . Permuting the indices of the covariant derivative (8) gives connection

$$\Gamma^{\eta}_{\mu\nu} = \{\}_{\mu\nu}^{\eta} + K^{\eta}_{\mu\nu}, \quad (10)$$

where the Christoffel symbol is

$$\{\}_{\mu\nu}^{\eta} \equiv \frac{1}{2}g^{\eta\rho}(g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}), \quad (11)$$

and the contorsion tensor is

$$K^{\eta}_{\mu\nu} \equiv \frac{1}{2}(Q_\mu\delta^{\eta}_{\nu} + Q_\nu\delta^{\eta}_{\mu} - Q^{\eta}g_{\mu\nu}). \quad (12)$$

Metric variation of the action (2) gives the metrical stress

$$8\pi\kappa^2 T_{\mu\nu} = \mathcal{A}G_{\mu\nu} - \mathcal{B}\Phi_\mu\Phi_\nu + \frac{1}{2}g_{\mu\nu}[\mathcal{B}\Phi_\rho^2 + V], \quad (13)$$

where $G_{\mu\nu}$ is the Einstein tensor. To get the stress expressed in terms of the Christoffel connection substitute (10) into the Riemann tensor (1)

$$R^{\alpha}_{\beta\nu\delta} - \bar{R}^{\alpha}_{\beta\nu\delta} = 2K^{\alpha}_{[\delta]R[\nu]}\Gamma^{\rho}_{\beta]R} + 2K^{\alpha}_{[\nu|\rho]}\Gamma^{\rho}_{\delta]R}, \quad (14)$$

substituting the object of non metricity (8) for the contorsion (12) the Riemann tensor becomes

$$R^{\alpha}_{\beta\nu\delta} - \bar{R}^{\alpha}_{\beta\nu\delta} = Q_{\beta[\gamma}\delta^{\alpha}_{\delta]} - Q^{\alpha}_{[\nu}g_{\delta]\beta} + \frac{1}{2}[Q_\beta Q_{[\delta}\delta^{\alpha}_{\nu]} + Q^{\alpha}Q_{[\gamma}g_{\delta]\beta} + Q^2_\rho g_{\beta[\gamma}\delta^{\alpha}_{\delta]}], \quad (15)$$

where the higher derivative terms use that Q is a gradient vector and that the connection is symmetric and $Q^2_\rho \equiv Q_\rho Q^\rho$. This is the same equation as equation (21)[22] with the assumption that Q is a gradient vector. From 15 or from

[25] section III§5 for connection (12) the Riemann tensor obeys the first, second, third symmetry identities and the Bianchi identity $\nabla_{[\omega} R_{\nu\mu]\lambda\rho} = 0$, note that Schouten [25] defines the Riemann tensor differently from (1) and that here the torsion vanishes. Contracting over α and γ gives the Ricci tensor

$$R_{\beta\gamma} - \bar{R}_{\beta\gamma} = \frac{d-2}{\alpha} \left[-2Q_{\beta\delta} + Q_{\beta}Q_{\delta} + g_{\beta\delta} \left(\frac{2}{\gamma-\alpha} \square Q - Q_{\rho}^2 \right) \right], \quad (16)$$

and then contracting over β and γ gives the Ricci scalar

$$R - \bar{R} = (1-d)\square Q + \frac{1}{\alpha}(1-d)(d-2)Q_{\rho}^2. \quad (17)$$

Using (16) and (17) the field equations (13) become

$$\frac{8\pi\kappa^2}{2} T_{\mu\nu} = \mathcal{A}\bar{G}_{\mu\nu} + \frac{(d-2)}{\alpha} \mathcal{A} \left(-2Q_{\mu\nu} + Q_{\mu}Q_{\nu} \right) - \mathcal{B}\Phi_{\mu}\Phi_{\nu} + \frac{1}{2} g_{\mu\nu} \left[(d-2)\mathcal{A}\square Q + \frac{\alpha(d-3)(d-2)}{4} \mathcal{A}Q_{\rho}^2 + \mathcal{B}\Phi_{\rho}^2 + V(\Phi) \right], \quad (18)$$

note the second derivative $Q_{\mu\nu}$ term. Using (9), Q can be eliminated

$$8\pi\kappa^2 T_{\mu\nu} = \mathcal{A}\bar{G}_{\mu\nu} - \mathcal{A}'\Phi_{\mu\nu} - \mathcal{B}_{\mathcal{A}}\Phi_{\mu}\Phi_{\nu} + \frac{1}{\gamma} g_{\mu\nu} [2\mathcal{A}'\square\Phi + (\mathcal{A}'' + \mathcal{B}_{\mathcal{A}})\Phi_{\rho}^2 + V] \quad (19)$$

where

$$\mathcal{B}_{\mathcal{A}} \equiv \mathcal{B} + \mathcal{A}'' + \frac{(1-d)\mathcal{A}}{\alpha(\alpha-\gamma)}. \quad (20)$$

Field variation gives the Euler equation

$$E(\{\}, \Phi, \mathcal{A}, \mathcal{B}, V) \equiv 2\mathcal{B}\square(\{\})\Phi + \mathcal{A}'R + \mathcal{B}'\Phi_{\rho}^2 - V' = 0, \quad (21)$$

in first term the covariant derivative is with the Christoffel connection, not the full connection as it comes from acting back on $\sqrt{-g}$. Using the Ricci identity stress conservation is

$$J^{\nu} \equiv 8\pi\kappa^2 T_{\mu\nu}^{;\mu} = -\frac{1}{\gamma} \Phi^{\nu} \left(E(\{\}, \Phi, \mathcal{A}, \mathcal{B}, V) + \frac{(1-d)}{\alpha(\alpha-\gamma)} E(\{\}, \Phi, 0, \frac{\mathcal{A}}{\alpha}, 0) \right), \quad (22)$$

the first term vanishes as it is just the field variation (21), vanishing current J^{ν} or in other words stress conservation implies that the second term produces a constraint on the form of the primary dilaton function \mathcal{A} . The two terms add linearly so that they can be replaced with one term with $\mathcal{B}_{new} = \mathcal{B} + \mathcal{A}'^2/\mathcal{A}$, however the two terms cannot be replaced by one term solely by changing the connection. One can work with the full connection throughout and produce $\nabla(\Gamma)_{\mu} T^{\mu\nu}$, but the situation is no better. There seems to be no way around it: either the stress is not conserved or there is a constraint on the primary dilaton function \mathcal{A} or the lagrangian has to be enlarged to include more terms. Once an object of non-metricity is given an effective mass can be calculated from equations of the same form as the Klein-Gordon equation and the Proca equation^[17,22], namely $(\square + m)g_{\mu\nu} = 0$, using equation (8) this gives a Euler equation in Q . In the notation (21) this equation is $E(\{\}, Q, 1, (2\epsilon + d - 4)Q, -mQ)$, where $\epsilon = 0, 1, 2, \text{ or } 3$ depending on the details of the calculation and also the primary dilaton function \mathcal{A} has been taken equal to one so that the corresponding lagrangian includes the Ricci scalar.

3. The Einstein Frame.

To transform to the Einstein frame the metric is rescaled by a conformal factor

$$g_{\mu\nu} \rightarrow \Omega g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \Omega^{-1} g^{\mu\nu}, \quad \sqrt{-g} \rightarrow \Omega^{\frac{d}{2}} \sqrt{-\bar{g}}. \quad (23)$$

The conformal factor connection is

$$L_{\mu\nu}^{\eta} \equiv \frac{1}{\gamma} \Omega^{-1} (\Omega_{\mu} \delta_{\nu}^{\eta} + \Omega_{\nu} \delta_{\mu}^{\eta} - \Omega^{\eta} g_{\mu\nu}). \quad (24)$$

and the connection transforms as

$$\{\} \rightarrow \bar{\{\}} + L, \quad \Gamma \rightarrow \bar{\Gamma} + L, \quad \text{with} \quad \bar{Q} = Q + \ln(\Omega). \quad (25)$$

For the lagrangian to be in the Einstein frame

$$\sqrt{-g} \mathcal{A} R \rightarrow (\Omega^{\frac{d}{2}} \sqrt{-\bar{g}}) \mathcal{A} (\Omega^{-1} R) = \sqrt{-\bar{g}} \bar{R}, \quad (26)$$

or

$$\Omega = \exp(-Q) = \mathcal{A}^{\frac{2-d}{2}}, \quad (27)$$

(9), (25) and (27) give $\bar{Q} = 0$, thus the transformed non-metricity vanishes, which is what would be expected if one started by Palatini varying in the Einstein frame.

The metric transformation (23) transforms the potential and the dilaton dynamical term to give the action

$$S = \int d^d x \sqrt{-\bar{g}} \left[\bar{R} - \frac{\mathcal{A}}{2} \Phi_{\rho}^2 - \mathcal{A}^{\frac{d(d-2)}{4}} V \right] + S_m[\exp(2\alpha(\Phi))g_{\mu\nu}, \psi_m]. \quad (28)$$

Defining

$$\Psi \equiv \int \frac{\mathcal{A}}{\mathcal{R}} d\Phi, \quad \bar{V} \equiv \mathcal{A}^{\frac{d(d-2)}{4}} V, \quad (29)$$

gives the Einstein frame action

$$S = \int d^d x \sqrt{-\bar{g}} [R - \Psi_\rho^2 - \bar{V}] + S_m[\exp(2\alpha(\Phi)) g_{\mu\nu}, \Psi_m], \quad (30)$$

where Φ in the S_m term can be replaced when the integral (29) has been evaluated, see below. Varying with respect to the metric \bar{g} gives the metrical stress

$$8\pi\kappa^2 \bar{T}_{\mu\nu} = \bar{G}_{\mu\nu} - \Psi_\mu \Psi_\nu + \frac{1}{\gamma} g_{\mu\nu} [\Psi_\rho^2 + \bar{V}]. \quad (31)$$

The matter action transforms as

$$S_m[\exp(2\alpha(\Phi)) g_{\mu\nu}, \Psi_m] \rightarrow S_m \left[\exp \left\{ 2\alpha \left(\int \frac{\mathcal{B}}{\mathcal{A}} d\Psi \right) \right\} \Omega \bar{g}_{\mu\nu}, \Psi_m \right], \quad (32)$$

with Ω given by (27). For an example in which the integral (29) can be evaluated choose

$$\mathcal{A} = \gamma \exp(c\Phi), \quad \mathcal{B} = \beta \exp(b\Phi), \quad (33)$$

with $\{c, b, \gamma, \beta\}$ constants. Then for $b = c$, $\Psi = \sqrt{\gamma/\beta} \Phi$ and for $b \neq c$

$$\Psi = \frac{z}{c-b} \left[\frac{\gamma}{\beta} \exp\left(\frac{(c-b)}{\gamma} \Phi\right) \right], \quad \Phi = \frac{z}{c-b} \ln \left(\frac{(c-b)}{\gamma} \left[\frac{\beta}{\gamma} \Psi \right] \right), \quad (34)$$

and

$$\Omega = \mathcal{A}^{\frac{(z-d)}{2}} = \gamma^{\frac{(z-d)}{2}} \exp\left(\frac{(z-d)}{\gamma} c\Phi\right), \quad (35)$$

substituting (35) into (32) gives

$$S_m \left[\gamma^{\frac{z-d}{2}} \exp\left(2\alpha(\Phi) + \frac{(z-d)}{\gamma} c\Phi\right) \bar{g}_{\mu\nu}, \Psi_m \right], \quad (36)$$

which simplifies to $S_m[\bar{g}_{\mu\nu}, \Psi_m]$ when

$$\alpha(\Phi) = \frac{(d-z)}{4} (c\Phi + \ln(\gamma)). \quad (37)$$

No transformation of the form $\Phi \rightarrow f(\Phi)$, will recover the second derivative $Q_{\mu\nu}$ term of (18); this can only be done by transforming the metric, $\bar{g}_{\mu\nu} \rightarrow \Omega^{-1} \bar{g}_{\mu\nu}$ which gives back (18) when $\bar{g} = g$.

In the Einstein frame there is no problem with stress conservation as

$$\bar{\nabla}^\nu \equiv 8\pi\kappa^2 \bar{T}_{\mu\nu}^{\mu\nu} = -\frac{1}{\gamma} \Psi^\nu \bar{E}(\{\}, \Psi, 0, 1, \bar{V}), \quad (38)$$

and in this case the Euler equation from varying the action (30) with respect to Ψ is just what appears on the right hand side. Transforming this Euler equation back to the Jordan frame

$$\bar{E}(\{\}, \Psi, 0, 1, \bar{V}) \rightarrow E(\{\} + C, \Phi, 0, \left[\frac{\mathcal{A}}{\mathcal{R}} \mathcal{A}^{\frac{(z-d)}{2}} \bar{V} \right]), \quad (39)$$

a contorsion C can be found which corrects for any Φ_ρ^2 term, in the present case the contorsion $C_{\mu\nu}^n$ is of the same form as (12), but with Q replaced by U and

$$U = \ln \left(\mathcal{A}^{\frac{(d^2-3d+3)}{2d}} B^{-\frac{1}{2d}} \right). \quad (40)$$

4. Cosmology.

The Robertson-Walker line element can be put in the form

$$ds^2 = -N(t)^2 dt^2 + R(t)^2 d\Sigma_{3,k}^2, \quad (41) \quad \text{where}$$

$$d\Sigma_{3,k}^2 = d\chi^2 + f(\chi)^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \quad (42)$$

and $f(\chi) = \{\sin(\chi), \chi, \sinh(\chi)\}$ for $k = +1, 0, -1$ respectively. N is called the lapse and R the scale factor. N can be absorbed into the line element in which case $N = 1$ and this gives the Robertson-Walker line element in proper time. For the choice $N = R$ Robertson-Walker space-time is conformal to the Einstein static universe and by convention the time coordinate is denoted by η . For $N = 1$ the scale factor R can be expanded as a Taylor series around a fixed time $t = t_0$ thus

$$R = R_0 \left[1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \mathcal{O}(t - t_0)^3 \right], \quad (43)$$

where the Hubble parameter and the deceleration parameter are defined by

$$H \equiv \dot{R}/R, \quad q \equiv -\ddot{R}R/\dot{R}^2, \quad (44)$$

the subscript "0" indicates that the parameter is evaluated at $t = t_0$, and $\dot{R} = \partial_t R$.

Take the pressure and the density to vanish so that there is vanishing stress which can be considered to be a

vacuum^[21]. This can be justified as whatever governs cosmological dynamics it appears not to be the pressure and density of luminous matter. This leaves just the scalar field and the metric, whether the scalar field can be called dark matter or dark field and so on is just a matter of terminology.

For the minimally coupled scalar-Einstein equations the Robertson-Walker line element (41) has solutions^[20]

$$\begin{aligned} k = 0, \quad \Xi &= \alpha\eta^{\frac{1}{2}}, \quad \Psi = \frac{\sqrt{3}}{2} \ln(\eta), \\ k = +1, \quad \Xi &= \alpha (\sin(\eta) \cos(\eta))^{\frac{1}{2}}, \quad \Psi = \frac{\sqrt{3}}{2} \ln(\tan(\eta)), \\ k = -1, \quad \Xi &= \alpha (\sinh(\eta) \cosh(\eta))^{\frac{1}{2}}, \quad \Psi = \frac{\sqrt{3}}{2} \ln(\tanh(\eta)), \end{aligned} \quad (45)$$

where

$$\alpha = 2R_0 \sqrt{H_0^2 R_0^2 / c^2 + k},$$

c is the speed of light and Ξ is equal to both the scale factor and the lapse, i.e. $\Xi = N = R$. The $k = 0$ solution is one of the few solutions known to have an exact form for the world function^[19]. To transfer to the Jordan frame, with $d = 4$, use

$$\bar{g}_{\mu\nu} \rightarrow \Omega^{-1} g_{\mu\nu} = \mathcal{A} g_{\mu\nu}, \quad \Phi = \int \sqrt{\frac{\mathcal{B}}{\mathcal{A}}} d\Psi, \quad (46)$$

so that, for the $k = 0$ example

$$R = N = \alpha \sqrt{\eta} \mathcal{A}, \quad (47)$$

and an arbitrary function has appeared in the scale factor R . Once the primary \mathcal{A} and secondary \mathcal{B} dilaton functions have been specified the solution can be transferred from conformal time to proper time, for example if

$$\mathcal{A} = \mathcal{B} = \exp(b\Phi), \quad (48)$$

$$\begin{aligned} & \text{[Trial mode] then with } b = 1/\sqrt{3} \\ \Phi = \Psi, \quad t &= \frac{\alpha}{2} \eta^2, \quad N = 1, \quad R = \sqrt{2\alpha t}, \end{aligned} \quad (49)$$

Another exact scalar-Einstein solution is^[17,18]

$$\begin{aligned} ds^2 &= -(1 + 2\sigma)dv^2 + 2dvdr + r(r - 2\sigma v) (d\theta^2 + \sin(\theta)^2 d\phi^2), \\ \Psi &= \frac{1}{2} \ln \left(1 - \frac{2\sigma v}{r} \right), \end{aligned} \quad (50)$$

where σ is a constant. This solution can implode from nothing to form a singularity of the Kretschmann curvature invariant $R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}$. It can also be transformed using (48) with $b = -2$

$$ds^2 = \frac{1}{1 - 2\sigma v/r} \left(-(1 + 2\sigma)dt^2 + 2dvdr \right) + r^2 (d\theta^2 + \sin(\theta)^2 d\phi^2), \quad (51)$$

showing that such implosions also happen in other theories. Non-metric quantum cosmology has been studied^[24].

5. Conclusion.

The Jordan frame action contains as much information as given by both the Einstein frame action and the transformation (27) between frames. This is not immediate as both the lagrangians and variables are different in the two cases. The Jordan frame action contains more information than the Einstein frame action; because all the information is contained in the action in the Jordan frame it must be preferable to the Einstein frame where there is both an action and an action independent transformation rule. Aesthetic grounds suggest that it is best to have just an action, as does Okham's razor. When the primary dilaton function \mathcal{A} is fixed, for example in many dilaton models it is fixed to be the exponential function, the extra information is still required, all that happens is that equations such as (9) and (27) simplify to (35). The transformation between the frames is not like a gauge transformation, because for a gauge transformation any result should be gauge independent. In other words for a gauge theory although the constraint might differ the dynamical information is the same, whereas here the Jordan frame is bigger as it contains the two freely specifiable dilaton functions \mathcal{A} and \mathcal{B} and so contains more dynamical information.

The Einstein frame has two advantages. The first is that stress conservation (38) is automatic, in other words the current vanishes $\bar{J}^\nu = 0$. For the Jordan frame in general there is a current (22), presumably this can be made to

vanish by adding more terms to the lagrangian. The second is that the Einstein frame is much simpler and easier to work with. This is illustrated by using known exact scalar-Einstein solutions and transferring them to the Jordan frame. For cosmology this gives equation (47) where the primary dilaton function \mathcal{A} has appeared in the scale factor: this suggests the possibility of comparing cosmological and particle physics predictions of what the primary dilaton function \mathcal{A} could be.

Palatini variation shows that if the dilaton exists then spacetime is non-metric, as conjectured in^[23]. One has essentially three choices either: i)Palatini variation does not correspond to anything in nature, ii)Spacetime is non-metric, iii)scalar-tensor theory does not correspond to anything in nature.

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